Jour. Ind. Soc. Ag. Statistics
49 (Golden Jub. No.), 1996-97 : 157–166

A Group-Theoretic Approach to Generalized Bahadur Expansion for Joint Probability Densities

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SUMMARY

Motivated by Bahadur [1] and Diaconis [3], a generalized Bahadur expansion for joint probability densities in a product of Borel spaces is introduced. When the marginal spaces are compact groups we exploit the techniques of harmonic analysis (Helson [4], Chandrasekharan [2]) to define the Bahadur correlations in such a way that they transform covariantly under all group translations.

Key words : Bahadur expansion, Bahadur correlation, Item analysis, Compact group, Representation.

1. *Introduction*

P. Diaconis [3] has amply demonstrated the usefulness of group representations in the spectral analysis of data. As a special application he has illustrated how Bahadur's item analysis [1] can be looked upon as the exploitation of harmonic analysis in the group $\mathbb{Z}_2^k$, the k-fold product of $\mathbb{Z}_2 = \{0, 1\}$ with addition modulo 2.

Here a general version of Bahadur's expansion is presented for a joint probability density in terms of orthonormal bases for the L^2-spaces of marginal distributions. This leads to natural notions of higher order interactions or correlations between the marginal components. Inspired by Diaconis' approach we present orthonormal bases for $L^2(P)$ where P is a general probability distribution in a compact group with a nowhere vanishing density. These bases are eminently suitable for writing Bahadur expansions in $\mathbb{Z}_d^k$ with $d \geq 2$. This reduces to Bahadur's method when $d = 2$. Furthermore the higher order correlations arising from these expansions transform covariantly under all group translations.

2. *A General Expansion for Joint Probability Densities*

Let $(X_i, \mathcal{F}_i)$, $i = 1, 2, \ldots, n$ be separable Borel spaces and let P be a probability measure on the product Borel space $\bigotimes_{i=1}^{n} (X_i, \mathcal{F}_i)$ with marginal distributions P_i on $(X_i, \mathcal{F}_i)$, $i = 1, 2, \ldots, n$. Denote by $Q = \bigotimes_{i=1}^{n} P_i$, the product measure and assume that P is absolutely continuous with respect to Q and

$$\int \left(\frac{dP}{dQ} \right)^2 dQ < \infty \tag{2.1}$$

Let $\{\mathbb{I}, \varphi_{i1}, \varphi_{i2}, \ldots\}$ be an orthonormal basis of functions for the Hilbert space $L^2(P_i)$ where $\mathbb{I}$ denotes the constant function identically equal to unity. Note that the sequence $\varphi_{ij}, j = 1, 2, \ldots$ may be of finite or infinite length depending on the nature of P_i. Define

$$\Psi_{i_1 i_2 \ldots i_m; j_1 j_2 \ldots j_m}(x_1, x_2, \ldots, x_n) = \prod_{r=1}^{m} \varphi_{i_r j_r}(x_{i_r}), \quad 1 \le i_1 < i_2 < \ldots i_m \le n \tag{2.2}$$

These functions together with $\mathbb{I}$ constitute an orthonormal basis for $L^2(Q)$ and by the assumption (2.1), $\frac{dP}{dQ}$ admits the Hilbert space expansion

$$\frac{dP}{dQ} = \mathbb{I} + \sum_{m=2}^{n} \sum_{1 \le i_1 < \ldots < i_m \le n} \sum_{j_1, j_2, \ldots, j_m} \rho_{i_1 i_2 \ldots i_m; j_1 \ldots j_m} \Psi_{i_1 i_2 \ldots i_m; j_1 \ldots j_m} \tag{2.3}$$

where

$$\begin{aligned} \rho_{i_1 i_2 \ldots i_m; j_1 \ldots j_m} &= \int \frac{dP}{dQ} \overline{\Psi}_{i_1 \ldots i_m; j_1 \ldots j_m} dQ \\ &= \mathbb{E}_P \overline{\Psi}_{i_1 \ldots i_m; j_1 \ldots j_m} \end{aligned} \tag{2.4}$$

$\mathbb{E}_P$ denoting expectation with respect to P. It is to be noted that

$$\rho_{i;j} = \mathbb{E}_P \overline{\Psi}_{ij} = \mathbb{E}_{P_i} \overline{\varphi}_{ij} = 0$$

and hence there is no term corresponding to $m = 1$ in (2.3). The right hand side of (2.3) converges in $L^2(Q)$ and

$$\int \left(\frac{dP}{dQ} - 1 \right)^2 dQ = \sum_{m=2}^{n} \sum_{1 \le i_1 < \ldots < i_m \le n} \sigma^2_{i_1 i_2 \ldots i_m} \tag{2.5}$$

where

$$\sigma^2_{i_1 i_2 \dots i_m} = \sum_{j_1, j_2, \dots, j_m} \left| \rho_{i_1 i_2 \dots i_m ; j_1 j_2 \dots j_m} \right|^2 \tag{2.6}$$

The left hand side of (2.5) measures the derivation of P from independence of its marginal constituents and $\sigma^2_{i_1 i_2 \dots i_m}$ is the contribution to this deviation arising from the m-th order interaction of the factors $i_1, i_2, \dots, i_m$. If $\sigma^2 = \int \left(\frac{dP}{dQ} - 1 \right)^2 dQ$ then the ratio $\sigma^2_{i_1 i_2 \dots i_m} / \sigma^2$ is a measure of the relative importance of the m-th order interaction of the factors $i_1, i_2, \dots, i_m$. It is also important to note that $\sigma^2_{i_1 i_2 \dots i_m}$ is independent of the choice of $\{ \varphi_{ij}, \ j = 1, 2, \dots; 1 \le i \le n \}$. This can be easily seen as follows : we have the direct sum decomposition

$$L^2(P_i) = \mathbb{C}\,\mathbb{I} \oplus \mathcal{K}_i, \quad \mathcal{K}_i = \mathbb{I}^{\perp}$$

In the Hilbert space tensor product

$$L^2(Q) = \bigotimes_{i=1}^{n} (\mathbb{C}\,\mathbb{I} \oplus \mathcal{K}_i)$$

the subspaces

$$\mathcal{K}_{i_1 i_2 \dots i_m} = \mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \mathcal{K}_{i_1} \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \mathcal{K}_{i_2} \otimes \dots \otimes \mathbb{I} \otimes \mathcal{K}_{i_m} \otimes \dots \otimes \mathbb{I}$$

where the unit vector $\mathbb{I}$ appears in positions other than $i_1 < i_2 < \dots < i_m$, make the direct sum decomposition

$$L^2(Q) = \mathbb{C}\,\mathbb{I} \oplus \bigoplus_{m=1}^{n} \bigoplus_{1 \le i_1 < \dots < i_m \le m} \mathcal{K}_{i_1 i_2 \dots i_m}$$

and

$$\sigma^2_{i_1 i_2 \dots i_m} = \| E_{i_1 i_2 \dots i_m} \left(\frac{dP}{dQ} - 1 \right) \|^2$$

where $E_{i_1 i_2 \dots i_m}$ is the projection on the subspace $\mathcal{K}_{i_1 i_2 \dots i_m}$

Suppose $\underline{x}^{(1)}, \underline{x}^{(2)}, \dots, \underline{x}^{(N)}$ are N independent observations with the probability law P in $\bigotimes_i X_i$. Define

$$\hat{\rho}_{i_1 i_2 \ldots i_m ; j_1 j_2 \ldots j_m} = \frac{1}{N} \sum_{r=1}^{N} \overline{\psi}_{i_1 \ldots i_m ; j_1 \ldots j_m} (\underline{x}^{(r)})$$

Owing to (2.4), $\hat{\rho}_{i_1 \ldots i_m ; j_1 \ldots i_m}$ has expectation $\rho_{i_1 \ldots i_m ; j_1 \ldots j_m}$. If $P = Q$ then as $N \to \infty$ the family $\{ \sqrt{N}\, \hat{\rho}_{i_1 \ldots i_m ; j_1 \ldots j_m} \}$ becomes a collection of independent (complex) Gaussian variables with mean 0 and unit variance in law.

Suppose that each X_i is a finite set of cardinality d_i and $P_i(\{ x \}) > 0$ for every $x \in X_i$. Assume that the set $\{ \varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{id_i} \}$ is closed under complex conjugation. Then under the law Q, the random variable

$$N \sum_{j_1 \ldots j_m} | \hat{\rho}_{i_1 \ldots i_m ; j_1 \ldots j_m} |^2$$

has, as $N \to \infty$, a limiting χ^2-distribution with $(d_{i_1} - 1)(d_{i_2} - 1) \ldots (d_{i_m} - 1)$ degrees of freedom for every $1 \le i_1 < i_2 < \ldots < i_m \le n$.

We call (2.3) a *Bahadur expansion* for $\frac{dP}{dQ}$. The (complex) scalar $\sigma^{-1} \rho_{i_1 i_2 \ldots i_m ; j_1 \ldots j_m}$ is called the m-th order *Bahadur correlation* between the $(i_1, \ldots, i_m)$-th marginal components arising from the basis elements $\varphi_{i_r j_r}$, $1 \le r \le m$. This is a brief summary of the central idea in Bahadur's item analysis cast in a general setting.

3. *Orthonormal Bases Arising from Harmonic Analysis in a Compact Group*

In Section 2 it is already seen, in the context of spectral analysis of data, the importance of constructing simple orthonormal bases of the form $\{ 1, \varphi_1, \varphi_2, \ldots \}$ in $L^2(P)$ where P is a probability measure on a separable Borel space $(X, \mathcal{F})$. Here, consider the case when X is a compact metric group and $\mathcal{F}$ is its Borel σ-algebra generated by the open subsets of X. To begin with we assume that X is an abelian group. Denote by $\hat{X}$ its character group. An element $\chi \in \hat{X}$ is a continuous homomorphism from X into the 1-dimensional torus, namely, the multiplicative group of complex numbers of modulus unity. Use the symbol dx to denote integration with respect to the normalised Haar measure on X. This normalised Haar measure is the uniform distribution on X. By $L^2(X)$ we mean the L^2-space with respect to this uniform distribution on X. The set $\hat{X}$ is at most countable and its elements constitute an orthonormal basis for $L^2(X)$. For any $f \in L^2(X)$ denote by $\hat{f}$ its "Fourier transform" which is a function defined on the character group $\hat{X}$ by

$$\hat{f}(\chi) = \int_X f(x)\, \chi(x)\, dx \tag{3.1}$$

The following are the basic relations :

$$f(x) = \sum_{\chi \in \hat{X}} \hat{f}(\chi)\, \overline{\chi(x)} \tag{3.2}$$

$$\int f(x)\, \overline{g(x)}\, dx = \sum_{\chi \in \hat{X}} \hat{f}(\chi)\, \overline{\hat{g}(\chi)} \tag{3.3}$$

for all $f, g \in L^2(X)$. Here the right hand side of (3.2) converges in $L^2(X)$ whereas the right hand side of (3.3) converges absolutely. (3.2) is the Fourier inversion formula whereas (3.3) is Parseval's identity (See Helson [4], Chandrasekharan [2]).

Our aim is to construct an orthonormal basis for $L^2(P)$ where P is a probability measure on X which has a density function $p(\cdot)$ with respect to the uniform distribution satisfying $p(x) > 0$ for every $x \in X$. Define

$$\xi_\chi(x) = \chi(x)\, p(x)^{-\frac{1}{2}}, \; x \in X, \chi \in \hat{X} \tag{3.4}$$

It is clear that

$$\mathbb{E}_P\, \xi_{\chi_1} \bar{\xi}_{\chi_2} = \delta_{\chi_1 \chi_2} \tag{3.5}$$

$\{ \xi_\chi, x \in \hat{X} \}$ is an orthonormal basis for $L^2(P)$ but the constant function $\mathbb{1}$ does not belong to this basis unless $p(\cdot) \equiv 1$. In order to overcome this obstacle introduce the functions

$$q(x) = p(x)^{\frac{1}{2}} \tag{3.6}$$

and

$$\eta_\chi(x) = \xi_\chi(x) - \hat{q}(\chi), \quad \chi \neq \mathbb{1} \tag{3.7}$$

$\hat{q}$ being the Fourier transform of the element $q \in L^2(X)$. Then we have

$$\mathbb{E}_P\, \eta_\chi = 0 \tag{3.8}$$

$$\mathbb{E}_P \eta_{\chi_1} \bar{\eta}_{\chi_2} = \delta_{\chi_1 \chi_2} - \hat{q}(\chi_1)\, \overline{\hat{q}(\chi_2)} \tag{3.9}$$

Then the family η_χ, $\chi \neq \mathbb{I}$ together with the constant function $\mathbb{I}$ span $L^2(P)$. We shall now orthonormalise $\{\eta_\chi, \chi \neq \mathbb{I}\}$ using (3.9). From (3.3) we have

$$\sum_{\chi \in \hat{X}} |\hat{q}(\chi)|^2 = \int q(x)^2 \, dx = 1$$

and hence

$$\sum_{\chi \neq 1} |\hat{q}(\chi)|^2 = 1 - |\hat{q}(\mathbb{I})|^2$$

$$= 1 - \left(\int q(x)\, dx\right)^2 \qquad (3.10)$$

Thus (3.9) can be expressed as

$$\mathbb{E}_P \eta_{\chi_1} \bar{\eta}_{\chi_2} = \delta_{\chi_1 \chi_2} - u(\chi_1)\overline{u(\chi_2)} + \alpha^2 u(\chi_1)\overline{u(\chi_2)} \qquad (3.11)$$

where

$$\alpha = \int_X q(x)\, dx \qquad (3.12)$$

$$u(\chi) = (1 - \alpha^2)^{-\frac{1}{2}} \hat{q}(\chi) \qquad (3.13)$$

$$\sum_{\chi \neq \mathbb{I}} |u(\chi)|^2 = 1 \qquad (3.14)$$

In other words the covariance matrix of the complex-valued random variables $\{\eta_\chi, \chi \neq \mathbb{I}, \chi \in \hat{X}\}$ undergoes a spectral decomposition with just two eigenvalues 1 and α^2 where the spectral projection corresponding to α^2 is one-dimensional. Now define

$$\varphi^P_\chi(x) = \eta_\chi(x) + (\alpha^{-1} - 1)\, u(\chi) \sum_{\chi' \neq 1} \overline{u(\chi')}\, \eta_{\chi'}(x) \qquad (3.15)$$

Then (3.11) and the preceding remark imply that the family $\varphi^P_\chi, \chi \neq \mathbb{I}$ together with the constant function $\mathbb{I}$ constitute an orthonormal basis for $L^2(P)$. Substituting for u from (3.13) and using (3.2) and (3.6) we observe that (3.15) simplifies to

$$\varphi^P_\chi(x) = \left\{ \chi(x) - \left(1 + \int_X q(y)\, dy\right)^{-1} \hat{q}(\chi) \right\} p(x)^{-\frac{1}{2}} - \left(1 + \int_X q(y)\, dy\right)^{-1} \hat{q}(\chi)$$

$$(3.16)$$

Thus the following theorem is proved.

Theorem 2.1. Let X be a compact metric abelian group with character group $\hat{X}$. Suppose P is a probability measure on X with density function $p(x) > 0$ for all $x \in X$ with respect to the uniform distribution. Define $q(x) = p(x)^{\frac{1}{2}}$ and $\varphi_\chi^P, \chi \neq 1, \chi \in \hat{X}$ by (3.16). Then the family of functions $\{\mathbb{I}, \varphi_\chi^P, \chi \neq \mathbb{I}\}$ is an orthonormal basis in $L^2(P)$.

Remark 2.2. Suppose that P is replaced by its left translate P_a defined by $P_a(E) = P(a^{-1} E)$ for any Borel set $E \subseteq X, a \in X$. Then (3.16) implies that

$$\varphi_\chi^{P_a}(x) = \chi(a)\, \varphi_\chi^P(a^{-1} x)$$

This has the following implication for the Bahadur correlations. If $X_i, i = 1, 2, \ldots, n$ are compact metric abelian groups and $\chi_i \in \hat{X}_i,\ i = 1, 2, \ldots, n$; P' is a probability measure on $X_1 \otimes \ldots \otimes X_n$ with marginals P'_i in X_i such that P' is absolutely continuous with respect to $Q = P'_1 \otimes \ldots \otimes P'_n$ and $\frac{dP'}{dQ'} \in L^2(Q')$ write

$$\rho^{P'}_{i_1 i_2 \ldots i_m;\, \chi_{i_1} \chi_{i_2} \cdots \chi_{i_m}} = \mathbb{E}_{P'} \overline{\Psi}_{i_1 i_2 \ldots i_m;\, \chi_{i_1} \chi_{i_2} \cdots \chi_{i_m}}$$

where

$$\Psi_{i_1 i_2 \ldots i_m;\, \chi_{i_1} \chi_{i_2} \cdots \chi_{i_m}}(x_1, \ldots, x_n) = \prod_{r=1}^{m} \varphi_{\chi_r}^{P'_{i_r}}(x_{i_r})$$

Then for $\underline{a} = (a_1, \ldots, a_n) \in X_1 \otimes \ldots \otimes X_n$ one has

$$\rho^{P'_{\underline{a}}}_{i_1 i_2 \ldots i_m;\, \chi_{i_1} \chi_{i_2} \cdots \chi_{i_m}} = \prod_{r=1}^{n} \overline{\chi_{i_r}(a_r)}\, \rho^{P}_{i_1 \ldots i_m;\, \chi_{i_1} \cdots \chi_{i_m}}$$

In other words the Bahadur correlations for $P'_{\underline{a}}$ and $P_{\underline{a}}$ differ just by the phase factors $\Pi_{r=1}^{m} \overline{\chi_{i_r}(a_r)}$ of modulus unity whenever we use orthonormal bases of Theorem 2.1.

Example 2.3. Let $X = \{0, 1, 2, ..., d-1\}$ with group operation being addition modulo d. Then $\hat{X} = \{\mathbb{I}, \chi_1, \chi_2, \cdots \chi_{d-1}\}$ where

$$\chi_k(j) = \exp 2\pi i k j / d, \qquad 0 \leq j \leq d-1$$

If $P(\{j\}) = p_j > 0$ for $0 \le j \le d-1$ is a probability distribution on X then define

$$\varphi_k^P(j) = \{\exp 2\pi ikj/d - (1+\alpha)^{-1}\hat{q}(k)\}\, p_j^{-\frac{1}{2}} - (1+\alpha)^{-1}\hat{q}(k) \quad (3.17)$$

where
$$\alpha = \frac{1}{d}\sum_{j=0}^{d-1} p_j^{1/2}$$

$$\hat{q}(k) = \frac{1}{d}\sum_{j=0}^{d-1} p_j^{1/2} \exp 2\pi ikj/d$$

Then $\{\mathbb{I}, \varphi_1^P, \varphi_2^P, \ldots, \varphi_{d-1}^P\}$ is an orthonormal basis for $L^2(P)$.

Going back to Section 2, if each X_i is the finite set $\{0, 1, 2, \ldots, d-1\}$ and P is the probability distribution on $X_1 \otimes \ldots \otimes X_n$ with i-th marginal P_i where $P_i(\{j\}) = p_{ij}$, $0 \le j \le d-1$ and $\varphi_{ik} = \varphi_k^{P_i}$ defined through (3.16) one obtains the Bahadur expansion (2.3). When $d = 2$ it coincides with the situation in item analysis.

Remark 2.4. Theorem 2.1 admits a simple generalization to the case when the compact group X is not necessarily abelian. Let $X = \{\pi_0, \pi_1, \ldots\}$ be a maximal family of inequivalent, irreducible unitary matrix representations of X with π_0 being the trivial one-dimensional representation so that $\pi_0(x) = 1$ for all $x \in X$. Denote by π an arbitrary element of $\hat{X}$ and $d(\pi)$ the order of the matrices $\pi(x)$, $x \in X$. Imitating (3.16) define the $d(\pi) \times d(\pi)$ matrix-valued function Φ_π^P by

$$\Phi_\pi^P(x) = d(\pi)^{\frac{1}{2}} \{(\pi(x) - (1+\alpha)^{-1}\hat{q}(\pi))\, p(x)^{-\frac{1}{2}} - (1+\alpha)^{-1}\hat{q}(\pi)\} \quad (3.18)$$

where $p(x)$ is the density of a probability measure P with respect to the normalised Haar measure (or uniform distribution on X), $p(x) > 0$ for all $x \in X$,

$$\alpha = \int_X p(x)^{\frac{1}{2}}\, dx$$

$$\hat{q}(\pi) = \int_X p(x)^{\frac{1}{2}}\, \pi(x) dx$$

and $\pi \ne \pi_0$. Then the constant function $\mathbb{I}$ together with the matrix entries $\varphi_{\pi,ij}^P(x)$, $1 \le i,j \le d(\pi)$ of Φ_π^P, $\pi \ne \pi_0$ constitute an orthonormal basis for $L^2(P)$.

If $X_i, 1 \le i \le n$ are compact metric groups, P' is a probability distribution on the product space $X_1 \otimes \ldots \otimes X_n$ with i-th marginal P_i', $1 \le i \le n$ so that condition (2.1) is fulfilled and each P_i' satisfies the conditions of the preceding paragraph then one can define the Bahadur correlations through

$$\rho^{P'}_{i_1 i_2 \ldots i_m;\ \pi^{(i_1)}_{j_1 k_1}\ \pi^{(i_2)}_{j_2 k_2} \ldots \pi^{(i_m)}_{j_m k_m}}$$

with the help of the bases $\{\mathbf{1}, \varphi^{P'_{r(r)}}_{\pi, j_r k_r}\}$ in $L^2(P'_r)$. If $P'_{\underline{a}}(E) = P'(\underline{a}^{-1} E)$ is the translated distribution through the element $\underline{a} \in X_1 \otimes \ldots \otimes X_n$ then

$$\rho^{P'_{\underline{a}}}_{i_1 i_2 \ldots i_m;\ \pi^{(i_1)}_{j_1 k_1}\ \pi^{(i_2)}_{j_2 k_2} \ldots \pi^{(i_m)}_{j_m k_m}}$$

$$= \sum_{s_1, s_2, \ldots, s_m} \pi^{(i_1)}_{j_1 s_1}(a_{i_1})\, \pi^{(i_2)}_{j_2 s_2}(a_{i_2}) \ldots \pi^{(i_m)}_{j_m s_m}(a_{i_m})\ \rho^{P'_{\underline{a}}}_{i_1 i_2 \ldots i_m;\ \pi^{(i_1)}_{j_1 k_1}\ \pi^{(i_2)}_{j_2 k_2} \ldots \pi^{(i_m)}_{j_m k_m}}$$

In other words the Bahadur correlations based on the factors $i_1, i_2, \ldots, i_m$ and the representations $\pi^{(i_r)}$, $1 \le r \le m$ of the groups X_{i_r}, $1 \le r \le m$ transform covariantly according to the tensor product representation $\otimes^m_{r=1} \pi^{(i_r)}(a_{i_r})$ under group translations by $\underline{a} \in X_1 \otimes \ldots \otimes X_m$.

Example 2.5. As an illustration for the orthonormal basis coming from (3.17) one may consider the group S_3, the symmetry group of all permutations of the set $\{1, 2, 3\}$. S_3 has six elements and three irreducible representations. One of them π_0 is trivial, the second one denoted by π_1 is the one-dimensional signature representation defined by $\pi_1(\sigma) = 1$ if σ is an even permutation and $\pi_1(\sigma) = -1$ if σ is an odd permutation and the third one π_2, the two dimensional representation with

$$\pi_2(\mathrm{id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pi_2(12) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\pi_2(13) = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad \pi_2(23) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\pi_2(123) = \begin{pmatrix} -\frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \pi_2(132) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

REFERENCES

[1] Bahadur, R.R., 1961. On classification based on responses to n dichotomous items. In : *Studies in Item Analysis* (H. Solomon, eds.), Stanford Univ. Press, Stanford, CA, U.S.A.

[2] Chandrasekharan, K., 1996. *A Course on Topological Groups,* Trim series No. 9, Hindustan Book Agency, Delhi.

[3] Diaconis, P., 1988. Group representations in probability and statistics, *IMS Lecture Notes - Monograph Series,* Hayward, CA, U.S.A.

[4] Helson, H., 1995. *Harmonic Analysis*, 2nd ed. Trim series No. 7, Hindustan Book Agency, Delhi.